ARKIV FÖR FYSIK Band 15 nr 25

Communicated 14 January 1959 by Oskar Klein and Ivar Waller

Transition matrix elements between states in a Coulomb field. II

By Per O. M. Olsson

Introduction

In a previous paper on the same subject [1] a simple integral occurring in the calculation of transition probabilities between states in a Coulomb field was studied to some extent. It was felt that the ideas presented in the paper could be generalized so that essentially the same methods could be applied to a very general class of matrix elements in a systematic way. There was some difficulty in deciding upon the most promising course to follow. The final result presented in [1] is propably very nearly as simple as is possible and it seems to be advisable to maintain the structure of the previous formula as much as possible. It is worth noting that this formula can be obtained from one of the power series expansions for the Coulomb wave functions which was given in a previous paper [2]. The convergent expansion is then obtained after term-wise integration. This simple procedure, however, does not easily lead to the desired result if the ordinary expressions for the wave functions are used. These functions do not seem to be sufficiently manageable and therefore a representation of the wave function in terms of one of the confluent Appell functions has been introduced. This representation has increased symmetry and allows a great number of expansions and transformations. As is well-known, a very general class of matrix elements occurring in physics can be expressed directly in terms of one of the hypergeometric functions of several variables introduced by Lauricella. Examples of such matrix elements are the radiative transition matrix elements between states in a Coulomb field. Absence of the Coulomb interaction is of course obtained as a special case provided that the interaction, which is then non-electromagnetic, can be expressed in suitable terms.

The introduction of the Appell function representation allows, as mentioned above, a great number of expansions and transformations. Such expansions and transformations lead to a great number of analytic continuations of the Lauricella function, in which the Lauricella function is, in general, expressed in terms of other hypergeometric functions. These expressions are as a rule complicated and it is also in many cases very difficult to decide whether the expansions are convergent or not.

It was strongly felt that, even in the more general cases when the matrix elements lead to a Lauricella function instead of an ordinary hypergeometric function as was the case in [1], the matrix elements could be expressed as a sum over terms connected by a recurrence relation in a similar way to that in the previous paper and that such an expansion of the matrix element should be convergent without modification for all values of the parameters which are of physical interest. For those values of the parameters involved for which the expansion given in [1] is too slowly convergent for practical purposes, the expansion can be easily modified so that the rapidity of the convergence is increased without changing the structure of the formula. This is of importance if an electronic computer is used. It is desirable that this property is maintained also for the more general types of matrix elements which may occur. This means that analytic continuations of the Lauricella functions should be given which are not only convergent, but also sufficiently rapidly convergent for practical purposes, for the whole domain of values of the arguments of the function which are of physical interest. Here, the selection of results of physical interest is of course only a tentative one.

Although the resulting formula for the matrix element obtained in [1] was not obtained from term-by-term integration after performing a series expansion of the integrand, it could, as was mentioned above, have been obtained in that way. When using such a procedure the convergence of the result depends on the choice of a proper expansion of the integrand. For the more general matrix elements that will be considered in this paper, the known expansions of the Lauricella function are not convergent or are very complicated whereas the expansion of the integrand is greatly simplified by the Appell function representation of the wave functions. The problem of finding useful expansions of the Lauricella function is then conveniently replaced by the problem of finding a proper expansion of a product of confluent Appell functions.

Since the simple power series expansion led to a satisfactory result for the integral studied in [1], it is reasonable to try the same method also for the more general types of matrix elements which will be discussed in this paper. In the following, it will be shown that the Lauricella function, and thus all matrix elements that may be expressed in terms of it, can be expanded in a series that is generally convergent and whose terms are connected by a recurrence relation. The number of coupled fields is not important but the recurrence relation connecting the terms of the expansion will become more complex and involve an increasing number of terms with an increasing number of coupled fields.

An expansion for products of fields and their matrix elements

The matrix elements that will be studied can be written

$$I = \int_0^\infty e^{-\mu r} r^{a-1} \prod_{i=1}^{N} \Phi_2(b_i, b_i'; c_i; x_i r, y_i r)\, dr, \tag{1}$$

where Φ_2 is a confluent Appell function defined by the convergent expansion

$$\Phi_2(b, b'; c; x, y) = \sum_{\substack{m=0\\n=0}}^{\infty} \frac{(b)_m (b')_n x^m y^n}{(c)_{m+n}\, m!\, n!}. \tag{2}$$

It seems to be sufficient to connect the pairs of parameters b_i and b_i' by the relation $b_i + b_i' = c_i$ in which case the Appell function is reduced to more elementary functions [3]:

$$\Phi_2(b,b';c;x,y) = e^y {}_1F_1(b;c;x-y). \tag{3}$$

From (3) follows the property

$$e^{\lambda}\Phi_2(b,b';b+b';x,y)=\Phi_2(b,b';b+b';x+\lambda,y+\lambda), \tag{4}$$

which means that we may put $\mu=0$ in (1) without loss of generality. We thus have a certain freedom of choice of the arguments $x_i r$ and $y_i r$ in (1) and, in fact, each of the many possible expansions of the Φ_2-functions with the restriction $b_i+b_i'=c_i$ will contain an arbitrary parameter having an influence on the convergence of the expansion. The integral (1), containing the reduced Φ_2-functions (3), has been given by A. Erdélyi [4]:

$$I=\int_0^\infty e^{-\mu r} r^{a-1}\prod_{i=1}^{N} e^{y_i r}\,{}_1F_1[b_i;\,c_i;\,(x_i-y_i)\,r]\,dr=$$

$$=\frac{\Gamma(a)}{\left(\mu-\sum\limits_{i=1}^{N} y_i\right)^a}F_A(a,b_1,\ldots,b_N;\,c_1\ldots c_N;\,\xi_1,\xi_2,\ldots\xi_N), \tag{5}$$

where

$$\xi_i=\frac{x_i-y_i}{\mu-\sum\limits_{i=1}^{N} y_i}$$

and F_A is one of the Lauricella functions of N variables. It is defined in the neighbourhood of the origin by the expansion

$$F_A=\sum_{n_1\cdots n_N}\frac{(a)_{n_1+\cdots+n_N}(b_1)_{n_1}(b_2)_{n_2}\cdots(b_N)_{n_N}}{(c_1)_{n_1}\cdots(c_N)_{n_N}\,n_1!\,n_2!\,\ldots\,n_N!}\,\xi_1^{n_1}\xi_2^{n_2}\ldots\xi_N^{n_N}. \tag{6}$$

It is not easy to find analytic continuations of this function beyond the domain of convergence which are of satisfactory simplicity. We therefore consider (5) as an integral representation of the F_A-function from which continuations will be derived. From the previous paper [1], where the special case $N=1$ was studied, we know that a satisfactory result can be obtained by expanding the Φ_2-function in powers of r. We shall therefore expand the N-fold product of Appell functions in the integrand of (1) in powers of r. In order to do this, we shall use an integral representation of the Φ_2-function [2]

$$\Phi_2(b,b';\,c;\,x,y)=$$

$$=\frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 e^{xu+y(1-u)}\,u^{b-1}(1-u)^{c-b-1}\,{}_1F_1[c-b-b';\,c-b;\,y\,(u-1)]\,du.$$

If we use throughout $b+b'=c$ the first parameter of the ${}_1F_1$-function is zero and thus

$$\Phi_2(b,b';\,b+b';\,x,y)=\frac{\Gamma(b+b')}{\Gamma(b)\,\Gamma(b')}\int_0^1 e^{xu+y(1-u)}\,u^{b-1}(1-u)^{b'-1}\,du. \tag{7}$$

The product of two such functions can then be written

$$\Phi_2(b_1, b_1'; b_1 + b_1'; x_1 r, y_1 r)\,\Phi_2(b_2, b_2'; b_2 + b_2'; x_2 r, y_2 r) =$$

$$= \frac{\Gamma(b_1 + b_1')}{\Gamma(b_1)\Gamma(b_1')} \int_0^1 e^{[x_1 u + y_1(1-u)]r}\,\Phi_2(b_2, b_2'; b_2 + b_2'; x_2 r, y_2 r)\, u^{b_1 - 1}(1-u)^{b_1' - 1}\, du =$$

$$= \frac{\Gamma(b_1 + b_1')}{\Gamma(b_1)\Gamma(b_1')} \int_0^1 u^{b_1 - 1}(1-u)^{b_1' - 1}\,\Phi_2[b_2, b_2'; b_2 + b_2'; (x_2 + x_1 u + y_1(1-u))r, (y_2 + x_1 u + y_1(1-u))r]\, du.$$

We have used here the transformation (4). The Φ_2-function can be expanded

$$\Phi_2[b_2, b_2'; b_2 + b_2'; (x_2 + x_1 u + y_1(1-u))r,\ (y_2 + x_1 u + y_1(1-u))r] =$$

$$= \sum_{n=0}^{\infty} \frac{(y_1 + y_2)^n \left[1 - \frac{y_1 - x_1}{y_1 + y_2} u\right]^n r^n}{n!}\, {}_2F_1\left(-n, b_2; b_2 + b_2'; \frac{\frac{y_2 - x_2}{y_1 + y_2}}{1 - \frac{y_1 - x_1}{y_1 + y_2} u}\right). \qquad (8)$$

This expansion is derived in [2]. Inserted into the previous integral it yields

$$\Phi_2(b_1, b_1'; b_1 + b_1'; x_1 r, y_1 r)\,\Phi_2(b_2, b_2'; b_2 + b_2'; x_2 r, y_2 r) =$$

$$= \frac{\Gamma(b_1 + b_1')}{\Gamma(b_1)\Gamma(b_1')} \sum_{n=0}^{\infty} \frac{[(y_1 + y_2)r]^n}{n!} \int_0^1 u^{b_1 - 1}(1-u)^{b_1' - 1}\left(1 - \frac{y_1 - x_1}{y_1 + y_1} u\right)^n \times$$

$$\times\, {}_2F_1\left(-n, b_2; b_2 + b_2'; \frac{\frac{y_2 - x_2}{y_1 + y_2}}{1 - \frac{y_1 - x_1}{y_1 + y_2} u}\right) du. \qquad (9)$$

But the F_A-function of two variables, which is usually called an Appell function and denoted F_2, has the integral representation [5]:

$$F_2(a, b, b'; c, c'; x, y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(c-b)\Gamma(b')\Gamma(c'-b')} \times$$

$$\times \int_0^1 \int_0^1 u^{b-1} v^{b'-1}(1-u)^{c-b-1}(1-v)^{c'-b'-1}(1 - ux - vy)^{-a}\, du\, dv.$$

Using an integral representation of the ordinary hypergeometric functions

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}\, du$$

the integration over u can be performed, yielding

$$F_2(a, b, b'; c, c'; x, y) = \\ = \frac{\Gamma(c')}{\Gamma(b')\,\Gamma(c'-b')} \int_0^1 v^{b'-1}(1-v)^{c'-b'-1}(1-vy)^{-a}\,{}_2F_1\left(a, b; c; \frac{x}{1-vy}\right) dv. \quad (10)$$

Comparing this integral representation of the F_2-function with the coefficients of the expansion (9), it is easily seen that the latter can be expressed in terms of the F_2-functions:

$$\Phi_2(b_1, b_1'; b_1 + b_1'; x_1 r, y_1 r)\,\Phi_2(b_2, b_2'; b_2 + b_2'; x_2 r, y_2 r) = \\ = \sum_{n=0}^{\infty} \frac{[y_1 + y_2)\,r]^n}{n!} F_2\left(-n, b_1, b_2; b_1 + b_1', b_2 + b_2'; \frac{y_1 - x_1}{y_1 + y_2}, \frac{y_2 - x_2}{y_1 + y_2}\right). \quad (11)$$

Since the first parameter in the F_2-function is a non-positive integer, the F_2-function is a polynomial in its arguments. It is further obvious that the expansion is convergent for all finite values of r since it must have the same radius of convergence as the Φ_2-functions.

It is now not difficult to see that the product of an arbitrary number of confluent Appell functions can be expanded in an analogous way:

$$\prod_{i=1}^{N} \Phi_2(b_i, b_i'; b_i + b_i'; x_i r, y_i r) = \\ = \sum_{n=0}^{\infty} \frac{[r \sum y_i]^n}{n!} F_A\left(-n, b_1, \dots b_N; c_1, \dots c_N; \frac{y_1 - x_1}{\sum y_i}, \dots \frac{y_N - x_N}{\sum y_i}\right), \quad (12)$$

where $$c_i = b_i + b_i'.$$

The restriction $b_i + b_i' = c_i$ is not necessary but is convenient. Similar expansions exist with the F_2-functions replaced by more general hypergeometric functions but such functions have not been studied. Since the task of deriving the necessary properties of the functions, which is probably laborious, has to precede their use, we will only consider the case $b_i + b_i' = c_i$.

The proof of (12) proceeds similarly to the proof for the product of two Appell functions. The integral representation (7) is introduced for all the Φ_2-functions but one and this latter function absorbs the exponential factors according to (4) before it is expanded. The coefficients will then be N − 1-fold integrals, in which one ${}_2F_1$-function will occur in the integrand. These integrals are integral representations of the Lauricella F_A-function of N variables in virtue of the integral representation [6]

$$F_A(a, b_1, b_2, \dots b_N; c_1, c_2 \dots c_N; x_1, x_2, \dots x_N) = \\ = \frac{\Gamma(c_1)\,\Gamma(c_2)\,\Gamma(c_N)}{\Gamma(b_1)\,\Gamma(b_2) \dots \Gamma(b_N)\,\Gamma(c_1 - b_1)\,\Gamma(c_2 - b_2) \dots \Gamma(c_N - b_N)} \times \\ \times \int_0^1 \int_0^1 \dots \int_0^1 u_1^{b_1-1} u_2^{b_2-1} \dots u_N^{b_N-1} (1-u_1)^{c_1-b_1-1} (1-u_2)^{c_2-b_2-1} \dots (1-u_N)^{c_N-b_N-1} \times \\ \times (1 - u_1 x_1 - u_2 x_2 - \dots - u_N x_N)^{-a}\, du_1 \dots du_N \quad (13)$$

which, with the aid of the integral representation of the ordinary hypergeometric functions, can be written

$$F_A(a, b_1, b_2, \dots b_N;\, c_1, c_2, \dots c_N;\, x_1, x_2, \dots x_N) =$$
$$= \frac{\Gamma(c_1)\Gamma(c_2)\dots\Gamma(c_{N-1})}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_{N-1})\Gamma(c_1-b_1)\Gamma(c_2-b_2)\dots\Gamma(c_{N-1}-b_{N-1})}\times$$
$$\times \int_0^1\int_0^1\dots\int_0^1 u_1^{b_1-1}u_2^{b_2-1}\dots u_{N-1}^{b_{N-1}-1}(1-u_1)^{c_1-b_1-1}(1-u_2)^{c_2-b_2-1}\dots(1-u_{N-1})^{c_{N-1}-b_{N-1}-1}\times$$
$$\times (1-u_1x_1-u_2x_2-\dots-u_{N-1}x_{N-1})^{-a}\,{}_2F_1\left(a, b_N;\, c_N;\, \frac{x_N}{1-\sum\limits_{i=1}^{N-1}u_i x_i}\right)du_1\,du_2\dots du_{N-1}.$$

Inserting the expansion derived into the integral (1) we obtain

$$I = \sum_{n=0}^{\infty}\frac{\left[\sum\limits_{i=1}^{N} y_i\right]^n}{n!}\times$$
$$\times F_A\left(-n, b_1, b_2, \dots b_N;\, b_1+b_1', b_2+b_2', \dots b_N+b_N';\, \frac{y_1-x_1}{\sum\limits_{i=1}^{N} y_i}, \dots \frac{y_N-x_N}{\sum\limits_{i=1}^{N} y_i}\right)\times$$
$$\times \int_0^{\infty} e^{-\mu r}\, r^{a+n-1}\, dr =$$
$$= \frac{\Gamma(a)}{\mu^a}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left[\frac{\sum\limits_{i=1}^{N} y_i}{\mu}\right]^n\times$$
$$\times F_A\left(-n, b_1, \dots b_N;\, b_1+b_1', \dots b_N+b_N';\, \frac{y_1-x_1}{\sum\limits_{i=1}^{N} y_i}, \dots \frac{y_N-x_N}{\sum\limits_{i=1}^{N} y_i}\right) \tag{14}$$

When it is convergent, this series gives the integral I. The convergence depends mainly on the values of the parameters x_i, y_i and μ. We now observe the possibility of adding arbitrary quantities to x_i, y_i and μ according to (4). This invariance of the integral I does not manifest itself in the F_A-function but affects the convergence of (14) and results in an arbitrariness in the expansion of the integral I. Using (5) we may express this circumstance as an expansion of the Lauricella function in which one free parameter occurs:

$$F_A\left(a, b_1, b_2, \dots b_N;\, c_1, c_2, \dots c_N;\, \frac{sz_1}{1+s}, \frac{sz_2}{1+s}, \dots \frac{sz_N}{1+s}\right) =$$
$$= (1+s)^a \sum_{n=0}^{\infty}\frac{(a)_n}{n!}(-s)^n F_A(-n, b_1, \dots b_N;\, c_1, \dots c_N;\, z_1, z_2, \dots z_N). \tag{15}$$

New parameters have been introduced in order to obtain a result that is more easily surveyed.

A great number of expansions of the Lauricella function can be derived starting from (12). As an example we derive an expansion containing products of Lauricella functions with fewer variables.

We have

$$\prod_{i=1}^{N} \Phi_2(b_i, b_i'; b_i+b_i'; x_i r, y_i r)$$

$$= \prod_{i=1}^{k} \Phi_2(b_i, b_i'; b_i+b_i'; x_i r, y_i r) \prod_{i=k+1}^{N} \Phi_2(b_i, b_i'; b_i+b_i'; x_i r, y_i r) =$$

$$= \sum_{m,n} \frac{r^{m+n}\left[\sum_{i=1}^{k} y_i\right]^m \left[\sum_{i=k+1}^{N} y_i\right]^n}{m!\,n!} \times$$

$$\times F_A\left(-m; b_1, \dots b_k; b_1+b_1', \dots b_k+b_k'; \frac{y_1-x_1}{\sum_{i=1}^{k} y_i}, \dots \frac{y_k-x_k}{\sum_{i=1}^{k} y_i}\right) \times$$

$$\times F_A\left(-n, b_{k+1}, \dots b_N; b_{k+1}+b_{k+1}', \dots b_N+b_N'; \frac{y_{k+1}-x_{k+1}}{\sum_{i=k+1}^{N} y_i}, \dots \frac{y_N-x_N}{\sum_{i=k+1}^{N} y_i}\right). \qquad (16)$$

Using this expansion instead of (12) we obtain for the integral (1):

$$I = \frac{\Gamma(a)}{\mu^a} \sum_{m,n} \frac{(a)_{m+n}}{m!\,n!} \left[\frac{\sum_{i=1}^{k} y_i}{\mu}\right]^m \left[\frac{\sum_{i=k+1}^{N} y_i}{\mu}\right]^n \times$$

$$\times F_A\left(-m, b_1, \dots b_k; b_1+b_1', \dots b_k+b_k'; \frac{y_1-x_1}{\sum_{i=1}^{k} y_i}, \dots \frac{y_k-x_k}{\sum_{i=1}^{k} y_i}\right) \times$$

$$\times F_A\left(-n, b_{k+1}, \dots b_N; b_{k+1}+b_{k+1}', \dots b_N+b_N'; \frac{y_{k+1}-x_{k+1}}{\sum_{i=k+1}^{N} y_i}, \dots \frac{y_N-x_N}{\sum_{i=k+1}^{N} y_i}\right) =$$

$$= \frac{\Gamma(a)}{\mu^a} \sum_{m} \frac{(a)_m}{m!} \left[\frac{\sum_{i=1}^{k} y_i}{\mu}\right]^m F_A\left(-m; b_1, \dots b_k; b_1+b_1', \dots b_k+b_k'; \frac{y_1-x_1}{\sum_{i=1}^{k} y_i}, \dots \frac{y_k-x_k}{\sum_{i=1}^{k} y_i}\right) \times$$

$$\times \sum_{n} \frac{(a+m)_n}{n!} \left[\frac{\sum_{i=k+1}^{N} y_i}{\mu}\right] \times$$

$$\times F_A\left(-n; b_{k+1}, \dots b_N; b_{k+1}+b_{k+1}', \dots b_N+b_N'; \frac{y_{k+1}-x_{k+1}}{\sum_{i=k+1}^{N} y_i}, \dots \frac{y_N-x_N}{\sum_{i=k+1}^{N} y_i}\right). \qquad (17)$$

From this expansion we obtain an expansion of the Lauricella function in a similar way to that in which (15) was derived

$$F_A\left(a, b_1, b_2 \dots b_N; c_1, c_2, \dots c_N; \frac{s z_1}{1+s+t}, \frac{s z_2}{1+s+t}, \dots \frac{s z_k}{1+s+t}, \frac{t z_{k+1}}{1+s+t}, \dots \frac{t z_N}{1+s+t}\right) =$$

$$= (1+s+t)^a \sum_{\substack{m=0\\ n=0}}^{\infty} \frac{(a)_{m+n}}{m!\,n!} (-s)^m (-t)^n F_A(-m, b_1, \dots b_k, c_1, \dots c_k; z_1, \dots z_k) \times$$

$$\times F_A(-n, b_{k+1}, \dots b_N; c_{k+1}, \dots c_N; z_{k+1}, \dots z_N). \quad (18)$$

The summation over n can be performed according to (15) and the result can be expressed in the following way:

$$F_A\left(a, b_1, b_2, \dots, b_N; c_1, c_2, \dots c_N; \frac{s z_1}{1+s}, \frac{s z_2}{1+s}, \dots \frac{s z_N}{1+s}\right) =$$

$$= (1+s)^a \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (-s)^n F_A(-n; b_1, b_2, \dots b_k; c_1, c_2, \dots c_k; z_1. z_2, \dots z_k) \times$$

$$\times F_A(a+n; b_{k+1}, b_{k+2}, \dots b_N; c_{k+1}, c_{k+2}, \dots c_N; s z_{k+1}, s z_{k+2}, \dots s z_N) \quad (19)$$

One parameter is lost during the summation.

A special case of (19) is obtained if we put $z_1 = z_2 = z_3 = \dots = z_k = 0$. For this case, we obtain an expansion analogous to (15)

$$F_A\left(a, b_1, b_2, \dots b_N; c_1, c_2, \dots c_N; \frac{z_1}{1+s}, \frac{z_2}{1+s}, \dots \frac{z_N}{1+s}\right) =$$

$$= (1+s)^a \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (-s)^n F_A(a+n, b_1, b_2, \dots b_N; c_1, c_2, \dots c_N; z_1, z_2, \dots z_N). \quad (20)$$

For $N = 1$, the formulas (15) and (20) are expansions of the ordinary hypergeometric functions. These special cases are found in Erdélyi, Magnus, Oberhettinger and Tricomi, *Higher Transcendental Functions* [7] and can be derived from certain bilinear relations between ordinary hypergeometric functions which have been given by Meixner [8]. Some of the relations given by Meixner follow from the expansions given above with $N = 2$ using an expansion of the F_2-function in terms of products of two ${}_2F_1$-functions, that has been given by Burchnall and Chaundy [9]. Erdélyi has also given a number of expansions of the Φ_2-function [10] from which a great number of other expansions of the Lauricella function may be derived. The expansions given here have been chosen because of their simplicity and because they are well suited for computations. By a proper choice of the free parameters in the expansions, it seems to be possible to obtain analytic continuations of the Lauricella function in a great number of cases of physical interest. The existence of recurrence relations for the F_A-functions makes a numerical computation very convenient and the computing time will be proportional to the number of terms which are used to approximate the F_A-function.

Applications

As a first application of the results derived in the previous section, we will consider the simple but instructive case when only one Φ_2-function occurs in the integrand:

$$I = \int_0^\infty e^{-r} r^{a-1} \Phi_2(\gamma + i\eta, \gamma - i\eta; 2\gamma; ipr, -ipr)\, dr. \tag{21}$$

This integral is a special case of an integral that was considered in a previous paper [1] which contained one Φ_2-function and for which a convergent expansion was derived without the restriction $b + b' = c$ that reduces the Φ_2-function to ordinary confluent hypergeometric functions. Because of the simplicity of the integral, it is well suited as a first example to which we can apply the previously derived expansions. It is also instructive to compare the following derivation with the result obtained in the previous paper.

Using the results of the previous section, we immediately obtain

$$I = \frac{\Gamma(a)}{(1+ip)^a}\, {}_2F_1\left(a, \gamma + i\eta; 2\gamma; \frac{2ip}{1+ip}\right) =$$

$$= \frac{\Gamma(a)}{(1+ip)^a} (1+s)^a \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (-s)^n\, {}_2F_1(-n; \gamma + i\eta; 2\gamma; z) \tag{22}$$

where
$$\frac{sz}{1+s} = \frac{2ip}{1+ip}.$$

We will assume that all the parameters occurring in (21) are real positive quantities except η which may be negative or zero. From the definition (2) of the Φ_2-function, it follows that it is a real function:

$$\Phi_2(\gamma + i\eta, \gamma - i\eta; 2\gamma; 2ipr, -ipr) = \Phi_2(\gamma - i\eta, \gamma + i\eta; 2\gamma; -ipr, ipr).$$

This means that we also have

$$I = \frac{\Gamma(a)}{(1-ip)^a}\, {}_2F_1\left(a, \gamma - i\eta; 2\gamma; \frac{-2ip}{1-ip}\right),$$

from which the singular nature of the integral for $ip = \pm 1$ is clearly evident. This can also be seen from the asymptotic behaviour of the integrand of the integral (21). It is then natural to choose the factor in front of the expansion (22) so that it takes account of the singular behaviour of the integral as much as possible. A good choice should be

$$1 + s = \frac{1}{1-ip}.$$

We then obtain

$$s = \frac{ip}{1-ip}, \qquad z = \frac{2}{1+ip},$$

and thus

$$I=\frac{\Gamma(a)}{(1+p^2)^a}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\left(\frac{-ip}{1-ip}\right)^n {}_2F_1\left(-n,\gamma+i\eta;2\gamma;\frac{2}{1+ip}\right). \tag{23}$$

This expansion converges for all real p-values. In order to prove this statement, it is sufficient to show that the ${}_2F_1$-function is absolutely bounded:

$$\left|{}_2F_1\left(-n,\gamma+i\eta;2\gamma;\frac{2}{1+ip}\right)\right|<A \quad \text{all } n.$$

This follows from the integral representation

$${}_2F_1\left(-n,\gamma+i\eta;2\gamma;\frac{2}{1+ip}\right)=\text{const}\int_0^1 u^{\gamma+i\eta-1}(1-u)^{\gamma-i\eta-1}\left(1-\frac{2u}{1+ip}\right)^n du,$$

since

$$\left|1-\frac{2u}{1+ip}\right|\leqslant 1, \quad 0\leqslant u\leqslant 1.$$

The above choice is not the only one possible for s, but it has the advantage of providing an expansion for which the convergence is easily proved as well as a natural method for obtaining the value of the parameter s.

The expansion (23) can be written

$$I=\frac{\Gamma(a)}{(1+p^2)^a}\sum_{n=0}^{\infty}c_n, \tag{24}$$

where the c_n are connected by a recurrence relation

$$c_{n+1}=\frac{a+n}{(2\gamma+n)(n+1)}\frac{p}{1+p^2}\{2[p(n+\gamma)-\eta]c_n-p(a+n-1)c_{n-1}\}, \tag{25}$$

which can be easily derived from the recurrence relations for the ${}_2F_1$-functions.

The Φ_2-function in (21) reduces according to (3) to confluent hypergeometric functions:

$$\Phi_2(\gamma+i\eta,\gamma-i\eta;2\gamma;ipr,-ipr)=e^{-ipr}{}_1F_1(\gamma+i\eta;2\gamma;2ipr).$$

Expansions like (24) for the integral (21) can be obtained without this restriction but, when the integrand contains more than one Φ_2-function, we shall consider functions with this restriction only, since a study of integrals over products of general Φ_2-functions seems to require the introduction of more general hypergeometric functions than the Lauricella and Appell functions.

As an example of a Φ_2-function with general parameters, we give an expansion in ${}_1F_1$-functions:

$$\Phi_2(a,b-a;c;x+y,y)=\sum_{n=0}^{\infty}\frac{(a)_n x^n}{(c)_n n!}{}_1F_1(b+n;c+n;y).$$

The function can also be expanded in cylindrical functions [2].

The integrand of (21) can thus contain arbitrary confluent hypergeometric functions as well as certain expansions in such functions and cylindrical functions. It is not our purpose to investigate to what extent the expansion (22) converges in such general cases. It will be considered sufficient that a convergent expansion can be derived in cases where the parameters and arguments have values that are representative for a great number of physical applications.

It is interesting to observe that also for large p-values the formula (22) allows an expansion that seems to be sufficiently rapidly convergent for practical purposes. This indicates a rather general applicability of the method.

In the limit $p \to \infty$, or if the exponential factor e^{-r} is not present in the integrand, we have to compute the ${}_2F_1$-function of argument 2:

$$ {}_2F_1(a, \gamma + i\eta; 2\gamma; 2) = (1+s)^a \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (-s)^n \, {}_2F_1(-n, \gamma + i\eta; 2\gamma; z) \tag{26} $$

with

$$ \frac{sz}{1+s} = 2. $$

We choose

$$ z = i\varrho, \quad \varrho \text{ real}, $$

then

$$ s = -\frac{2}{2 - i\varrho}. $$

If $|\varrho| < 1$ it can be shown that [11]:

$$ |{}_2F_1(-n, \gamma + i\eta; 2\gamma; i\varrho)| = 0 \left[\frac{1}{(|\varrho| n)^{\gamma}} \right] $$

which proves the convergence of (26).

As a second application we shall consider the integral:

$$ I = \int_0^{\infty} e^{-r} r^{a-1} \prod_{k=1}^{2} \Phi_2(b_k, b_k'; b_k + b_k'; i p_k r, -i p_k r) \, dr. \tag{27} $$

We obtain

$$ \begin{aligned} I &= \frac{\Gamma(a)}{[1 + i(p_1 + p_2)]^a} F_2\left(a, b_1, b_2; b_1 + b_1'; b_2 + b_2'; \frac{2ip_1}{1 + i(p_1 + p_2)}, \frac{2ip_2}{1 + i(p_1 + p_2)}\right) \\ &= \frac{\Gamma(a)}{[1 + i(p_1 + p_2)]^a} (1+s)^a \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (-s)^n F_2(-n, b_1, b_2; b_1 + b_1'; b_2 + b_2'; z_1, z_2), \end{aligned} \tag{28} $$

where

$$ \frac{sz_1}{1+s} = \frac{2ip_1}{1 + i(p_1 + p_2)}, \qquad \frac{sz_2}{1+s} = \frac{2ip_2}{1 + i(p_1 + p_2)}. $$

Both the Φ_2-functions in (27) reduce to ordinary confluent hypergeometric functions according to (3). From the asymptotic behaviour of the ${}_1F_1$-function for a large argument, the singular nature of the integral for $i(p_1 \pm p_2) = \pm 1$ can be easily seen. We therefore put

$$\frac{1+s}{1+i(p_1+p_2)} = \frac{1}{[1+i(p_1+p_2)]\,[1+i(p_1-p_2)]\,[1-i(p_1+p_2)]\,[1-i(p_1-p_2)]} = \frac{1}{1+h} \tag{29}$$

which gives
$$s = -\frac{(p_1-p_2)^2 - i(p_1+p_2)\,[1+(p_1-p_2)^2]}{1+(p_1-p_2)^2 - i(p_1+p_2)\,[1+(p_1-p_2)^2]}. \tag{30}$$

Thus $|s| < 1$ for real, finite values of p_1 and p_2.

Further
$$\left.\begin{aligned} z_1 &= \frac{2\,i\,p_1}{i(p_1+p_2)-h}, \\ z_2 &= \frac{2\,i\,p_2}{i(p_1+p_2)-h}. \end{aligned}\right\} \tag{31}$$

For these values of z_1 and z_2, F_2-function in the expansion (28) is absolutely bounded:

$$|F_2(-n, b_1, b_2; b_1+b_1'; b_2+b_2'; z_1, z_2)| < A, \quad \text{all } n.$$

This can be shown from the integral representation of the F_A-function given in the previous section,

$$F_2(-n, b_1, b_2; b_1+b_1'; b_2+b_2'; z_1, z_2)$$
$$= \text{const} \int_0^1 \int_0^1 u_1^{b_1-1}\, u_2^{b_2-1}\, (1-u_1)^{b_1'-1}\, (1-u_2)^{b_2'-1}\, (1-u_1 z_1 - u_2 z_2)^n \, d\,u_1\, d\,u_2$$

since
$$|1-u_1 z_1 - u_2 z_2| = \left| \frac{h - i\,p_1(1-2\,u_1) - i\,p_2(1-2\,u_2)}{h - i\,(p_1+p_2)} \right| \leqslant 1$$

for $0 \leqslant u_1 \leqslant 1$, $0 \leqslant u_2 \leqslant 1$ and p_1 and p_2 real and positive which makes h real as a consequence of (29). For the sake of simplicity we assume that the parameters a, b_i and b_i' have such values that the integral representation used for the F_2-function actually has a meaning.

The procedure outlined above can obviously be applied to integrals containing arbitrarily many Φ_2-functions, but the rapidity of the convergence seems to decrease with an increasing number of Φ_2-functions in the product for the given choice of s. In addition, the recurrence relations for the F_A-functions in the coefficients of the expansions will become more complicated and the number of functions connected by the recurrence relation will increase. For the integral (27), where F_2-functions occur in the terms of its expansion, no more than five terms are connected by a recurrence relation since it is known that five F_2-functions, where the parameters differ by integers, are connected by a linear relation in which the coefficients are polynomials in the arguments.

The recurrence relation for the F_2-function with integer steps in the first parameter does not seem to have been derived. It is, however, possible to avoid the difficulty of obtaining recursion relations for the F_A-functions by using expansions into products of F_4-functions with fewer variables, such as (19). As an example we may consider the expansion (28). Since it is absolutely convergent it follows that the use of the expansion (19) with the same value for the parameter s will also be convergent, since the series may be summed in an arbitrary manner. Instead of using the recurrence relation for the F_2-function we can then use the simpler recurrence relations for the ${}_2F_1$-function, with the unimportant complication of using it twice for each term.

The general nature of the integrals investigated in this paper makes it practically impossible to state to what extent the given expansions can be used to compute the integrals. From the two applications above, it is obvious, however, that simple and rapid methods for computing integrals of the type studied can be obtained in a great number of cases of physical interest.

ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to Professor O. Klein for facilitating the work on this paper in many ways and for his kind encouragement. By means of stimulating discussions on many related topics, Drs. I. Fischer-Hjalmars, B. Laurent, B. Nagel and U. Uhlhorn have contributed to the author's investigation of this field.

The financial assistance of the Swedish Atomic Committee is greatly acknowledged.

Institute of Theoretical Physics, University of Stockholm.

REFERENCES

1. P. O. M. Olsson, Arkiv f. Fysik *15*, 159 (1959).
2. P. O. M. Olsson, Arkiv f. Fysik *15*, 131 (1959).
3. J. L. Burchnall and T. W. Chaundy, Quart. J. of Math. *12*, 112 (1941).
4. A. Erdélyi, Math. Z. *42*, 125 (1936).
5. A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, Higher Transcendental Functions, Vol. I, p. 230. McGraw-Hill Book Company Inc., New York, 1953.
6. P. Appell and J. Kampé de Fériet, Fonctions hypergéometriques et hypersphériques. Gauthier-Villars, Paris, 1926.
7. A. Erdélyi *et al.*, Higher Transcendental Functions, Vol. I, p. 85. McGraw-Hill Book Company Inc., New York, 1953.
8. J. Meixner, Deutsche Math. *6*, 341 (1941).
9. J. L. Burchnall and T. W. Chaundy, Quart. J. of Math. *12*, 112 (1941).
10. A. Erdélyi, Proc. Roy, Soc. Edinburgh, *LIX*, 224 (1939).
11. A. Erdélyi, *et al.*, Higher Transcendental Functions, Vol. I, p. 77. McGraw-Hill Book Company Inc., New York, 1953.

Tryckt den 22 april 1959

Uppsala 1959. Almqvist & Wiksells Boktryckeri AB